NOTATION

$T(r, x, \tau)$, temperature at any point of the experimental object; r, x, current coordinates of object; τ, time; $\Theta(r, x, \tau) = T(r, x, \tau) - T_0$, excess temperature of experimental object; $q(\tau)$, heat-flux density; $U(x, t)$, potential in cross section x of long line; $i(x, t)$, current in cross section x of long line; x, current coordinate of long line; t, time; γ, constant of propagation; p, Laplace-transformation parameter; z_W, wave resistance of RC structure; z_{lo}, load resistance; A, B, matrix elements of quadrupole; $h_1(t)$, $h_2(t)$, transition characteristics of the models; k, tunable coefficient; erfc x = 1 − erf x; $\text{erf}\,x = 2/\sqrt{\pi}\int_0^x e^{-x^2}dx$, Gaussian error function.

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FORMULAS FOR THE DISCREPANCY GRADIENT IN THE ITERATIVE SOLUTION OF INVERSE HEAT-CONDUCTION PROBLEMS. II. DETERMINING THE GRADIENT IN TERMS OF A CONJUGATE VARIABLE

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UDC 536.24

The construction of the functional-deficiency gradient is considered for the iterative solution of inverse problems in the case of an equation of parabolic type. Nonlinear formulations of the problem are considered in the general case.

In the first part of this report [1], formulas were obtained for the discrepancy gradient in terms of the Green's function of the corresponding boundary problem. A more general method of finding the gradient is based on solving the conjugate boundary problem [2, 3]. Below, an approach to deriving the conditions of this problem and formulas for the discrepancy gradient allowing a rigorous basis for the results obtained to be established is outlined.

Suppose that in a region with mobile boundaries $Q_\tau = \{X_1(\tau) < x < X_2(\tau),\ 0 < \tau < \tau_m\}$ a quasilinear parabolic equation is specified

$$CT_\tau = (\lambda T_x)_x + KT_x + g. \tag{1}$$

The initial and boundary conditions for Eq. (1) are

$$T(x, 0) = \xi(x), \tag{2}$$

$$[\alpha_i \lambda T_x + \beta_i T]_{x=X_i(\tau)} = p_i(\tau), \quad i = 1, 2, \tag{3}$$

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Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 52, No. 4, pp. 668-675, April, 1987. Original article submitted January 6, 1986.

where the numbers α_i, β_i define a particular type of boundary conditions.

In determining $\xi(x)$, $p_1(\tau)$, $p_2(\tau)$, i.e., in solving the retrospective and boundary inverse problems, the coefficients and free term in Eq. (1) will be assumed, in the general case, to be functions of $T(x, \tau)$, x, and τ: $C(T, x, \tau)$ $\lambda(T, x, \tau)$, $K(T, x, \tau)$, $g(T, x, \tau)$. In solving the coefficient inverse problems, one or more of the functions $C(T)$, $\lambda(T)$, $K(T)$, $g(T)$ are the desired quantities.

Suppose that, as an additional condition, the dependence $f(\tau) = T(d(\tau), \tau)$ is known at one spatial point $x = d(\tau)$ moving over time. The coordinate of this point corresponds to the conditions: $X_1(\tau) \leq d(\tau)$ when $\alpha_1 \neq 0$, $X_1(\tau) < d(\tau)$ when $\alpha_1 = 0$, $d(\tau) \leq X_2(\tau)$ when $\alpha_2 \neq 0$, $d(\tau) < X_2(\tau)$ when $\alpha_2 = 0$. Regarding the curves of $X_1(\tau)$, $X_2(\tau)$, $d(\tau_2)$, $\tau \in [0, \tau_m]$, it is assumed that they are piecewise-smooth functions.

The generalization to the case of N dependences $f_n(\tau) = T(d_n(\tau), \tau)$, $n = \overline{1, N}$, $N \geq 1$ will be given in the third part of this report.

To solve this problem using one of the gradient methods, it is necessary to know the gradient of the discrepancy functional

$$J(\bar{u}) = \frac{1}{2} \int_0^{\tau_m} [T(\bar{u}, d(\tau), \tau) - f(\tau)]^2 d\tau \tag{4}$$

in terms of the vector function

$$\bar{u} = \{\xi(x), p_1(\tau), p_2(\tau), \lambda(T), C(T), K(T), g(T)\}, \tag{5}$$

where $\xi(x) \in L_2[X_1(0), X_2(0)]$; $p_i(\tau)$, $f(\tau) \in L_2[0, \tau_m]$; $\lambda(T)$, $C(T)$, $K(T)$, $g(T) \in L_2[T_0, T_M]$. Here T_0, T_M are the boundaries of the region in which the corresponding coefficients of the free term of Eq. (1) must be found.

Below, two auxiliary boundary problems are required: the problem for the field increment $T(x, \tau)$ and its conjugate problem.

Problem for the Field Increment

Suppose that the components of $\bar{u}$ undergo an increment: $C(T) + \Delta C(T)$, $\lambda(T) + \Delta\lambda(T)$, $K(T) + \Delta K(T)$, $g(T) + \Delta g(T)$, $\xi(x) + \Delta\xi(x)$, $p_1(\tau) + \Delta p_1(\tau)$, $p_2(\tau) + \Delta p_2(\tau)$. This new set of data will correspond to the function $T(x, \tau) + v(x, \tau)$ and the system in Eqs. (1)-(3) is written in the form

$$[C(T+v) + \Delta C(T+v)][T+v]_\tau = \{[\lambda(T+v) + \Delta\lambda(T+v)][T+v]_x\}_x + [K(T+v) + \Delta K(T+v)][T+v]_x + g(T+v) + \Delta g(T+v), \quad (x, \tau) \in Q_\tau; \tag{6}$$

$$T(x, 0) + v(x, 0) = \xi(x) + \Delta\xi(x); \tag{7}$$

$$\{\alpha_i [\lambda(T+v) + \Delta\lambda(T+v)][T+v]_x + \beta_i [T+v]\}_{x=X_i(\tau)} = p_i(\tau) + \Delta p_i(\tau), \quad i = 1, 2. \tag{8}$$

Here the dependences of C, λ, K, g on x and τ, which may be present in solving the retrospective and boundary inverse problems, are conventionally not shown.

The corresponding conditions in Eqs. (1)-(3) are now subtracted from Eqs. (6)-(8) and $C(T + v)$, $\lambda(T + v)$, $K(T + v)$, $g(T + v)$ are taken in the form of Taylor-series expansions, retaining their first two terms, while $\Delta C(T + v)$, $\Delta\lambda(T + v)$, $\Delta K(T + v)$, $\Delta g(T + v)$ are taken in Taylor-series expansions retaining the first term. In addition, it is taken into account that the coefficients C, λ, K, g in the direct problem in Eqs. (1)-(3) are known functions of the temperature $T(x, \tau)$, which is regarded as specified in the given conditions for the increment $v(x, \tau)$. Therefore, below, these coefficients are known functions only of the independent variables x and τ. As a result, after several transformations, the following formulation of the problem is obtained for determining $v(x, \tau)$

$$v_\tau = a_1 v_{xx} + a_2 v_x + a_3 v + q + \omega, \quad (x, \tau) \in Q_\tau; \tag{9}$$

$$v(x, 0) = \Delta\xi; \tag{10}$$

$$B_{i\tau} v(X_i(\tau), \tau) \equiv [\gamma_i v_x + \sigma_i v]_{x=X_i(\tau)} = \rho_i(\tau) + \omega_i, \quad i = 1, 2, \tag{11}$$

where $a_1 = \lambda/C$; $a_2 = (2\lambda_x + K)/C$; $a_3 = (\lambda_{xx} + g_T + K_x - C_\tau)/C$; $q = (1/C) \times [\Delta\lambda T_{xx} + (\Delta\lambda)_x \cdot T_x + \Delta K T_x + \Delta g - \Delta C T_\tau]$; $\gamma_i = \alpha_i \lambda (X_i(\tau), \tau)$, $\sigma_i = \alpha_i \lambda_x (X_i(\tau), \tau) + \beta_i$; $\rho_i(\tau) = \Delta p_i(\tau) - \alpha_i \Delta\lambda(X_i(\tau), \tau) T_x(X_i(\tau), \tau)$; ω, ω_1, ω_2 are the remainder terms (if $\alpha_1 = \alpha_2 = 0$, then $\omega_1 = \omega_2 = 0$).

Using the substitution $y = (x - X_1(\tau))/(X_2(\tau) - X_1(\tau))$, $t = \tau$, the region Q_τ is transformed into the rectangle $R = (0, 1) \times (0, \tau_m)$. Here $a_i(x, \tau)$ and $q(x, \tau)$ are transformed to $\bar{a}_i(y, t)$ and $\bar{q}(y, t)$, respectively, and the formulation of Eqs. (9)-(11) takes the form

$$v_t = b_1 v_{yy} + b_2 v_y + b_3 v + \bar{q} + \bar{\omega}, \quad (y, t) \in R; \tag{12}$$

$$v(y, 0) = \Delta\bar{\xi}(y); \tag{13}$$

$$B_{1t} v(0, t) \equiv [l_1 v_y + \sigma_1 v]_{y=0} = \rho_1(t) + \bar{\omega}_1; \tag{14}$$

$$B_{2t} v(1, t) \equiv [l_2 v_y + \sigma_2 v]_{y=1} = \rho_2(t) + \bar{\omega}_2, \tag{15}$$

where

$$b_1 = \frac{\bar{a}_1(y, t)}{[X_2(t) - X_1(t)]^2}; \quad b_2 = \frac{1}{X_2(t) - X_1(t)} \{\bar{a}_2(y, t) + X_1'(t) + y[X_2'(t) - X_1'(t)]\}, \quad X_i' \equiv \frac{dX_i}{dt};$$

$$b_3 = \bar{a}_3(y, t); \quad l_1 = \frac{\gamma_1(t)}{X_2(t) - X_1(t)}; \quad l_2 = \frac{\gamma_2(t)}{X_2(t) - X_1(t)}.$$

Thus, eliminating $\bar{\omega}$, $\bar{\omega}_1$ $\bar{\omega}_2$, i.e., proceeding to a linear approximation of the boundary problem for the increment $v(y, t)$, the system in Eqs. (12)-(15) is analogous to the formulation of Eqs. (2)-(4) in [1].

Moreover, using the results of [4], it may be shown that the problem obtained from Eqs. (9)-(11) and correspondingly Eqs. (12)-(15) by discarding nonlinear terms is the Freshe derivative A' of the operator A corresponding to the desired value of the function $T(d(\tau), \tau)$. To this end, it is sufficient to require that $C(T)$, $K(T)$, and $g(T)$ are functions that are twice continuously differentiable, while $\lambda(T)$ is triply continuously differentiable; $p_i \in W_2^1[0, \tau_m]$, $i = 1, 2$; $\xi \in W_2^1\ [X_1(0), X_2(0)]$ and the matching conditions of the initial and boundary conditions are satisfied.

Below, it is assumed that nonlinear terms are discarded in Eqs. (9)-(11) and (12)-(15).

Considering the case of boundary conditions of the second and third kinds ($\alpha_1 \neq 0$, $\alpha_2 \neq 0$), the problem in Eqs. (12)-(15) is reduced to zero boundary conditions

$$v_t = b_1 v_{yy} + b_2 v_y + b_3 v + \bar{q} + \bar{z}, \quad (y, t) \in R; \tag{16}$$

$$v(y, 0) = 0; \tag{17}$$

$$[l_1 v_y + \sigma_1 v]_{y=0} = 0; \tag{18}$$

$$[l_2 v_y + \sigma_2 v]_{y=1} = 0. \tag{19}$$

To determine the function $\bar{z} = \bar{z}(y, t)$, the solutions of Eqs. (12)-(15) and (16)-(19) are written in terms of Green's functions – see Eq. (25) in [1] – and equated

$$\int_0^1 \Delta\bar{\xi}(y') G(y, t; y', 0)\, dy' - \int_0^t \frac{b_1(0, t')}{l_1(t')} [\Delta p_1(t')] G(y, t; 0, t')\, dt' +$$

$$+ \int_0^t \frac{b_1(1, t')}{l_2(t')} [\Delta p_2(t')] G(y, t; 1, t')\, dt' = \int_0^t dt' \int_0^1 [\bar{z}(y', t')] G(y, t; y', t')\, dy',$$

and hence

$$\bar{z}(y, t) = -\frac{b_1(y, t)}{l_1(t)} [\rho_1(t)] \delta(y) + \frac{b_1(y, t)}{l_2(t)} [\rho_2(t)] \delta(y - 1) + \Delta\bar{\xi}\delta(t),$$

where $\delta(\cdot)$ is a delta function.

With this function $\bar{z}(y, t)$, the problem in Eqs. (16)-(19) is identical to the problem in Eqs. (12)-(15) and hence that in Eqs. (9)-(11). Making the inverse transition (from y, t to x, τ) in Eqs. (16)-(19), a formulation equivalent to Eqs. (9)-(11) is found, taking account of the properties of the δ function: $\delta(a\zeta) = \delta(\zeta)/|a|$, when $a \neq 0$ is real

$$v_\tau = a_1 v_{xx} + a_2 v_x + a_3 v + q + z, \quad (x, \tau) \in Q_\tau; \tag{20}$$

$$v(x, 0) = 0; \tag{21}$$

$$B_{i\tau} v(X_i(\tau), \tau) = 0, \quad i = 1, 2, \tag{22}$$

where

$$z(x, \tau) = -\frac{a_1}{\gamma_1}(\rho_1)\,\delta(x - X_1(\tau)) + \frac{a_i}{\gamma_2}(\rho_2)\,\delta(x - X_2(\tau)) + \Delta\xi\delta(\tau).$$

<u>Conjugate Problem</u>

Supposing that

$$L = \frac{\partial}{\partial\tau} - A_{x\tau}, \quad A_{x\tau} = a_1\frac{\partial^2}{\partial x^2} + a_2\frac{\partial}{\partial x} + a_3,$$

$$\chi = q - a_1\frac{\rho_1}{\gamma_1}\,\delta(x - X_1(\tau)) + a_1\frac{\rho_2}{\gamma_2}\,\delta(x - X_2(\tau)) + \Delta\xi\delta(\tau),$$

the problem in Eqs. (20)-(22) is rewritten in the form

$$Lv = \chi, \quad v \in D_L. \tag{23}$$

Here $D_L = \{v \in G(Q_\tau);\ B_{i\tau} v(X_i(\tau), \tau) = 0,\ i = 1, 2;\ v(x, 0) = 0\}$ is the region of definition of operator L; $G(Q_\tau)$ is the set of functions $v = v(x, \tau)$ with continuous derivatives v_τ, v_x, v_{xx} in Q_τ.

Considering L as an operator mapping $D_L \subset L_2(Q_\tau)$ in space L_2, the scalar product of the elements Lv, $\psi \in L_2$ is written

$$(Lv, \psi) = \iint_{Q_\tau} \psi L v\, dx\, d\tau,$$

where $\psi = \psi(x, \tau)$ is a function sufficiently smooth in $\bar{Q}_\tau$.

The conjugate operator L^* is defined in accordance with the Lagrangian identity

$$(Lv, \psi) = (v, L^*\psi) \tag{24}$$

and the scalar product on the left is expanded as follows

$$\iint_{Q_\tau} \psi L v\, dx\, d\tau = \iint_{Q_\tau} \psi(v_\tau - a_1 v_{xx} - a_2 v_x - a_3 v)\, dx\, d\tau = I_1 - I_2,$$

where

$$I_1 = \iint_{Q_\tau} [-v\psi_\tau + (a_1\psi)_x v_x + (a_2\psi)_x v - a_3\psi v]\, dx\, d\tau,$$

$$I_2 = \iint_{Q_\tau} [(a_1\psi v_x + a_2\psi v)_x - (\psi v)_\tau]\, dx\, d\tau.$$

Formulas for integration by parts will be required below [4]; for the present case, they take the form

$$\iint_{Q_\tau} u_x v\, dx\, d\tau = -\iint_{Q_\tau} u v_x\, dx\, d\tau + \int_{\partial Q_\tau^+} uv\, d\tau, \quad u,\ v \in W_2^{1,0}(Q_\tau),$$

$$\iint_{Q_\tau} u_\tau v\, dx\, d\tau = -\iint_{Q_\tau} u v_\tau\, dx\, d\tau - \int_{\partial Q_\tau^+} uv\, dx, \quad u,\ v \in W_2^1(Q_\tau),$$

where ∂Q_τ^+ is the boundary of the region Q_τ, oriented so that on passing around the boundary the region remains on the left. In the present case

$$\int_{\partial Q_\tau^+} uv d\tau = \int_0^{\tau_m} uv\Big|_{x=X_2(\tau)} d\tau - \int_0^{\tau_m} uv\Big|_{x=X_1(\tau)} d\tau,$$

$$\int_{\partial Q_\tau^+} uv dx = \int_0^{\tau_m} uv\Big|_{x=X_2(\tau)} X_2'(\tau)\,d\tau - \int_{X_1(\tau_m)}^{X_2(\tau_m)} uv\Big|_{\tau=\tau_m} dx - \int_0^{\tau_m} uv\Big|_{x=X_1(\tau)} X_1'(\tau)\,d\tau + \int_{X_1(0)}^{X_2(0)} uv\Big|_{\tau=0} dx.$$

Then

$$\iint_{Q_\tau} (a_1\psi)_x v_x dx d\tau = -\iint_{Q_\tau} (a_1\psi)_{xx} v dx d\tau + \int_0^{\tau_m} (a_1\psi)_x v\Big|_{X_1(\tau)}^{X_2(\tau)} d\tau.$$

Thus

$$I_1 = \iint_{Q_\tau} v[-\psi_\tau - (a_1\psi)_{xx} + (a_2\psi)_x - a_3\psi]\,dx d\tau + \int_0^{\tau_m} v\,(a_1\psi)_x\Big|_{x=X_1(\tau)}^{x=X_2(\tau)} d\tau.$$

Using the same formulas for integration by parts, integral I_2 is found to be

$$I_2 = \int_0^{\tau_m} (a_1\psi v_x + a_2\psi v)\Big|_{X_1(\tau)}^{X_2(\tau)} d\tau - \int_0^{\tau_m} \psi v\Big|_{x=X_2(\tau)} X_2'(\tau)\,d\tau + \int_0^{\tau_m} \psi v\Big|_{x=X_1(\tau)} X_1'(\tau)\,d\tau + \int_{X_1(\tau_m)}^{X_2(\tau_m)} \psi v\Big|_{\tau=\tau_m} dx - \int_{X_1(0)}^{X_2(0)} \psi v\Big|_{\tau=0} dx.$$

The latter integral is zero, since $v(x, 0) = 0$.

Invoking the boundary conditions in Eq. (22), the final expression for I_2 is

$$I_2 = \int_0^{\tau_m} \left\{ v\psi\left[X_2' + a_2 - \frac{\sigma_2}{\gamma_2} a_1\right]_{x=X_2(\tau)} - v\psi\left[X_1' + a_2 - \frac{\sigma_1}{\gamma_1} a_1\right]_{x=X_1(\tau)} \right\} d\tau + \int_{X_1(\tau_m)}^{X_2(\tau_m)} \psi(x, \tau_m)\, v(x, \tau_m)\, dx.$$

Turning to the identity in Eq. (24), it is found that $\psi(x, \tau)$ must satisfy the conditions of the following boundary problem, which is conjugate to the problem in Eqs. (20)-(22)

$$L^*\psi(x, \tau) = \zeta(x, \tau), \quad (x, \tau) \in Q_\tau; \tag{25}$$

$$\psi(x, \tau_m) = 0; \tag{26}$$

$$B_{i\tau}^*\psi\Big|_{x=X_i(\tau)} \equiv \left[(a_1\psi)_x - \psi\left(X_i' + a_2 - \frac{\sigma_i}{\gamma_i} a_1\right)\right] = 0, \quad i = 1,\ 2, \tag{27}$$

where $L^* = -\partial/\partial\tau - A_{x\tau}^*$, $A_{x\tau}^*\psi = (a_1\psi)_{xx} - (a_2\psi)_x + a_3\psi$. The function $\zeta(x, \tau)$ will be found below.

Formulas for the Discrepancy-Functional Gradient

The problem in Eqs. (25)-(27) is written in a form analogous to Eq. (23)

$$L^*\psi = \zeta, \quad \psi \in D_{L^*},$$

where $D_{L^*} = \{\psi \in G(Q_\tau);\ B_{i\tau}^*\psi|_{x=X_i(\tau)} = 0,\ i = 1, 2;\ \psi(x, \tau_m) = 0\}$.

Suppose that L^{-1} and L^{*-1} are operators inverse to L and L^*, respectively. The scalar product for the elements v and ψ is

$$(v, \psi) = (LL^{-1}v, \psi) = (L^{-1}v, L^*\psi) = (v, L^{-1*}L^*\psi).$$

Hence $L^{-1*}L^* = I$ in D_{L^*} (I is a unit operator). Thus

$$L^{-1*} = L^{*-1}. \tag{28}$$

For the Freshe-differentiable operator A of the direct problem, the gradient of the functional J is found from the formula $J' = (A')^*(A\bar{u} - f)$ and the increment in the functional ΔJ takes the form

$$\Delta J \equiv J(\bar{u} + \Delta\bar{u}) - J(\bar{u}) = (J', \Delta\bar{u}) + o(\|\Delta\bar{u}\|) = (A\bar{u} - f, A'\Delta\bar{u}) + o(\|\Delta\bar{u}\|),$$

or in the adopted notation

$$\Delta J = \int_0^{\tau_m} [T(\bar{u},\ d(\tau),\ \tau) - f(\tau)]\, v(d(\tau),\ \tau)\, d\tau + o(\|\Delta\bar{u}\|).$$

Next ΔJ is written in terms of a δ function, in the form

$$\Delta J = \int_0^{\tau_m} d\tau \int_{X_1(\tau)}^{X_2(\tau)} v(x,\ \tau)\, h(\tau)\, \delta(x - d(\tau))\, dx + o(\|\Delta\bar{u}\|),$$

where $h(\tau) = T(\bar{u},\ d(\tau),\ \tau) - f(\tau)$.

Noting that $v(x,\ \tau) = L^{-1}(\chi)$ and introducing the conjugate operator, it is found that

$$\Delta J = \iint_{Q_\tau} L^{-1}(\chi)\, h(\tau)\, \delta(x - d(\tau))\, dx d\tau + o(\|\Delta\bar{u}\|) = (\chi,\ L^{-1*}[h(\tau)\, \delta(x - d(\tau))]) + o(\|\Delta\bar{u}\|). \quad (29)$$

Since Eq. (28) holds, it follows that

$$L^{-1*}[h(\tau)\, \delta(x - d(\tau))] = \psi,$$

where $\psi = \psi(x,\ \tau)$ is the solution of the problem in Eqs. (25)-(27), where $\zeta(x,\ \tau) = h(\tau) \cdot \delta(x - d(\tau))$.

Expanding the function χ in Eq. (29) gives

$$\Delta J = \left(\psi,\ q - \frac{a_1}{\gamma_1} \rho_1 \delta(x - X_1) + \frac{a_1}{\gamma_2} \rho_2 \delta(x - X_2) + \Delta\xi\delta(\tau)\right) + o(\|\Delta\bar{u}\|). \quad (30)$$

Establishing a relation between the direct and conjugate problems, which leads ultimately to the derivation of Eq. (30), is also necessary in a rigorous formulation, most notably in terms of the applicability of the formulas for integration by parts, which imposes the requirement of definite smoothness on the integrand. The conditions on the initial data of the direct problem ensuring the required smoothness of solutions of the direct and conjugate problems may be obtained using results of [5]. For this purpose, it is sufficient to require satisfaction of the conditions ensuring that the operator of the direct problem is differentiable.

From Eq. (30), ΔJ is found in the form

$$\Delta J = (\Phi,\ \Delta\bar{u}) + o(\|\Delta\bar{u}\|).$$

It is obvious here that $\Phi = J'$.

To obtain the component of the gradient J'_u with respect to a single component u of vector $\bar{u}$, all the components of $\bar{u}$ except for that required must be set equal to zero in Eq. (30). Then it follows from Eq. (30) that

$$\Delta J = (\Phi_u,\ \Delta u) + o(\|\Delta u\|),\ \text{i.e.}\ \Phi_u = J'_u.$$

Thus, formulas for the components of the gradient with respect to ξ, p_1, p_2 are obtained from Eq. (30)

$$J'_\xi(x) = \psi(x,\ 0),\quad J'_{p_1}(\tau) = -\psi(X_1(\tau),\ \tau)\, \frac{a_1(X_1(\tau),\ \tau)}{\gamma_1(\tau)},$$

$$J'_{p_2}(\tau) = \psi(X_2(\tau),\ \tau)\, \frac{a_1(X_2(\tau),\ \tau)}{\gamma_2(\tau)}.$$

Determining the components of the gradient J'_λ, J'_C, J'_K, J'_g is a considerably more complicated problem. It may be simplified by seeking the coefficients and free term of Eq. (1) in parameterized form (see [6], for example)

$$\lambda(T) = \sum_{j=1}^{M_1} \lambda_j \varphi_j(T), \qquad C(T) = \sum_{j=1}^{M_2} C_j \varphi_j(T),$$

$$K(T) = \sum_{j=1}^{M_3} K_j \varphi_j(T), \qquad g(T) = \sum_{j=1}^{M_4} g_j \varphi_j(T),$$

where $\{\phi_j\}_1^M$ is a specified system of basis functions; λ_j, C_j, K_j, g_j are numerical coefficients, which remain to be determined.

The increments of these functions are written analogously:

$$\Delta\lambda(T)=\sum_{j=1}^{M_1}\Delta\lambda_j\varphi_j(T),\quad \Delta C(T)=\sum_{j=1}^{M_2}\Delta C_j\varphi_j(T), \tag{31}$$

$$\Delta K(T)=\sum_{j=1}^{M_3}\Delta K_j\varphi_j(T),\quad \Delta g(T)=\sum_{j=1}^{M_4}\Delta g_j\varphi_j(T). \tag{32}$$

Expanding q and ρ_i in Eq. (30), taking account of the approximation in Eqs. (31) and (32), and performing simple transformations, formulas are obtained for the gradients of the functional in Eq. (4) with respect to the vectors $\bar{\lambda} = \{\lambda_j\}_1^{M_1}$, $\bar{C} = \{C_j\}_1^{M_2}$, $\bar{K} = \{K_j\}_1^{M_3}$, $\bar{g} = \{g_j\}_1^{M_4}$, respectively

$$J'_{\lambda_l}=\Phi_0+\Phi_1+\Phi_2,\quad l=\overline{1, M_1},$$

where

$$\Phi_0=\int_0^{\tau_m}d\tau\int_{X_1(\tau)}^{X_2(\tau)}\frac{\psi(x,\tau)}{C(x,\tau)}\left[T_{xx}(x,\tau)\varphi_l(T(x,\tau))+T_x^2(x,\tau)\frac{d\varphi_l(T(x,\tau))}{dT}\right]dx,$$

$$\Phi_i=\int_0^{\tau_m}\frac{\psi(x,\tau)}{C(x,\tau)}T_x(x,\tau)\varphi_l(T(x,\tau))\Big|_{x=X_i(\tau)}d\tau,\quad i=1,2;$$

$$J'_{C_l}=-\int_0^{\tau_m}d\tau\int_{X_1(\tau)}^{X_2(\tau)}\frac{\psi(x,\tau)}{C(x,\tau)}T_\tau(x,\tau)\varphi_l(T(x,\tau))\,dx,\quad l=\overline{1, M_2};$$

$$J'_{K_l}=\int_0^{\tau_m}d\tau\int_{X_1(\tau)}^{X_2(\tau)}\frac{\psi(x,\tau)}{C(x,\tau)}T_x(x,\tau)\varphi_l(T(x,\tau))\,dx,\quad l=\overline{1, M_3};$$

$$J'_{g_l}=\int_0^{\tau_m}d\tau\int_{X_1(\tau)}^{X_2(\tau)}\frac{\psi(x,\tau)}{C(x,\tau)}\varphi_l(T(x,\tau))\,dx,\quad l=\overline{1, M_4}.$$

Knowing $J'_\xi(x)$, $J'_{p_i}(\tau)$, $J'_{\bar{\lambda}}$, $J'_{\bar{C}}$, $J'_{\bar{K}}$, $J'_{\bar{g}}$, the search for the corresponding quantities may be organized on the basis of regularizing gradient algorithms [2, 3, 6, 7].

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